MATH 3060 Tutorial 5

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In this tutorial, we discussed some Midterm questions.

1 Some definitions

1. Let d, d' be two metrics on X, we say d is stronger than d' if $d' \leq Cd$ for some $C \in \mathbb{R}$.
2. A map $f : (X, d) \to (X', d')$ is an isometric embedding if for any $x, x' \in X$, $d'(f(x), f(x')) = d(x, x')$.
3. A metric space is COMPLETE if every Cauchy sequence has a limit.

2 Questions of this tutorials

True or False

(a) Let $f : (X, d) \to (X, d')$ be a bijective continuous map such that f^{-1} is also continuous, then (X, d) is complete if and only if (X', d') is complete.
Ans: False, consider $\mathbb{R}$ and $(0, 1)$.

(b) Suppose d, d' be two metrics on X, with d stronger than d'. If $U \subset X$ is open with respect to d', U is also open with respect to d.
Ans: True

(c) Suppose d, d' be two metrics on X, with d stronger than d'. If $U \subset X$ is open with respect to d, U is also open with respect to d'.
Ans: False

(d) Suppose d, d' be two metrics on X. Suppose also that a subset $U \subset X$ is open with respect to d if and only if U is also open with respect to d', then d and d' are equivalent.
Ans: False, consider d and $d/(1 + d)$

(e) Let (X, d) be a metric space, and $(C(X), d_\infty)$ be the set of continuous functions on X with metric defined by $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$. Then $C(X)$ is complete.
Ans: True

(f) $C([0,1])$ equipped with the $L1$ metric is complete.

(g) Any isometric embedding is injective.
Ans: True

(h) Let (X, d) be a metric space. For any $x \in X$, we can define $d_x \in C(X)$ by $d_x(y) = d(x, y)$. The map $x \mapsto d_x$ is an isometric embedding of X into $C(X)$.
Ans: False. Construct a sequence of functions f_n with $f_n(x) = 1$ on $[0, \frac{1}{2}]$, and $\int_{\frac{1}{2}}^{1} |f(x)| dx \to 0$.

(i) Every metric space has a countable dense subset.
Ans: False, consider a discrete metric.